

The f -vector of the descent polytope

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Abstract

For a positive integer n and a subset $S \subseteq [n-1]$, the descent polytope DP_S is the set of points (x_1, \dots, x_n) in the n -dimensional unit cube $[0, 1]^n$ such that $x_i \geq x_{i+1}$ if $i \in S$ and $x_i \leq x_{i+1}$ otherwise. We discuss several ways to compute the f -vector (f_0, f_1, \dots, f_n) of this polytope. First, we express the f -polynomial $F_S(t) = f_0 + f_1 t + \dots + f_n t^n$ as a sum over all subsets of $[n-1]$. Second, we use certain factorizations of the associated word over a two-letter alphabet to describe the f -polynomial $F_S(t)$. Finally, we give efficient recursions to compute $F_S(t)$. We show that the f -vector is maximized when the set S is the alternating set $\{1, 3, 5, \dots\} \cap [n-1]$. We also derive a generating function for $F_S(t)$, written as a formal power series in two non-commuting variables with coefficients in $\mathbb{Z}[t]$.

1 Introduction

For a set $S \subseteq [n-1] = \{1, 2, \dots, n-1\}$, define the *descent polytope* DP_S to be the set of points (x_1, \dots, x_n) in \mathbb{R}^n such that $0 \leq x_i \leq 1$, and

$$\begin{cases} x_i \geq x_{i+1} & \text{if } i \in S, \\ x_i \leq x_{i+1} & \text{if } i \notin S. \end{cases}$$

Thus DP_S is the *order polytope* of the ribbon poset $Z_S = \{z_1, z_2, \dots, z_n\}$ defined by the cover relations $z_i \cdot > z_{i+1}$ if $i \in S$ and $z_i < \cdot z_{i+1}$ if $i \notin S$; see [9]. Descent polytopes occur as a subdivision of the n -dimensional unit cube in the recent work [2] of Ehrenborg, Kitaev, and Perry. It is clear that the set S and its complement $\bar{S} = [n-1] - S$ yields the same descent polytope up to an affine transformation.

In this paper our primary goal is to compute the f -vector of the descent polytope DP_S . Recall that for an n -dimensional polytope, the f -vector is the integer vector (f_0, f_1, \dots, f_n) , where f_i is the number of i -dimensional faces in the polytope. Observe that $f_n = 1$ since the descent polytope is a face of itself. For $S \subseteq [n-1]$, define the f -polynomial of the descent polytope DP_S to be

$$F_S(t) := \sum_{i=0}^n f_i \cdot t^i.$$

To simplify notation, we will often write F_S instead of $F_S(t)$. As we show in Section 2, F_S can be expressed as a sum of polynomials taken over all subsets of S .

The volume of the descent polytope DP_S is given by $\beta(S)/n!$, where $\beta(S)$ denotes the number of permutations in the symmetric group \mathfrak{S}_n with descent set S . A classical result in combinatorics is

that $\beta(S)$ is maximized when S is an alternating set, that is, $S = \{1, 3, 5, \dots\} \cap [n-1]$, or equivalently $S = \{2, 4, 6, \dots\} \cap [n-1]$; see [1, 6, 7, 8, 11]. We show that the same result holds for the f -vector of the descent polytope. That is, the number of i -dimensional faces of the descent polytope DP_S is maximized when S is the alternating set.

A different way to encode subsets of the set $[n-1]$ is by a word in letters \mathbf{x} and \mathbf{y} of length $n-1$. Viewing \mathbf{x} and \mathbf{y} as non-commutative variables and the words as monomials, we consider the non-commutative generating function $\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}} \cdot \mathbf{v}$. We determine this generating function, which turns out to be a rational function. Using this rational function we obtain a more concise expression for the f -polynomial of the descent polytope DP_S . A second way to encode subsets of $[n-1]$ is by compositions. In Section 4 we use this encoding to obtain more recurrences to compute the f -polynomial F_S .

We end the paper with a few open questions and directions for further research.

2 An expression for the f -polynomial $F_{\mathbf{v}}$

Let \mathbf{x} and \mathbf{y} be two non-commuting variables. For $S \subseteq [m]$, define $\mathbf{v}_S = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_m$ where

$$\mathbf{v}_i = \begin{cases} \mathbf{x} & \text{if } i \notin S, \\ \mathbf{y} & \text{if } i \in S. \end{cases}$$

We denote F_S by $F_{\mathbf{v}_S}$; such notation has an advantage, as \mathbf{v}_S encodes not only $S \subseteq [n-1]$ but also the dimension n . Since pairs $(n, S \subseteq [n-1])$ are in bijective correspondence with \mathbf{xy} -words via $S \mapsto \mathbf{v}_S$, it is natural to parameterize the f -polynomials of descent polytopes by \mathbf{xy} -words and write $F_{\mathbf{v}}$, where $\mathbf{v} = \mathbf{v}_S$ for some $S \subseteq [|\mathbf{v}|]$, and $|\mathbf{v}|$ denotes the length of the word \mathbf{v} .

For a \mathbf{xy} -word $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{n-1}$, define the statistic $\kappa(\mathbf{v})$ by $\kappa(\mathbf{v}) = 2 + |\{i : \mathbf{v}_i \neq \mathbf{v}_{i+1}\}|$ for $\mathbf{v} \neq 1$, and $\kappa(1) = 1$. A direct observation is the number of facets of the descent polytope is described by κ .

Lemma 2.1. *The number of $(n-1)$ -dimensional faces of the n -dimensional descent polytope $\text{DP}_{\mathbf{v}}$ is given by*

$$f_{n-1}(\text{DP}_{\mathbf{v}}) = n - 1 + \kappa(\mathbf{v}).$$

Proof. There are $n-1$ supporting hyperplanes of the form $x_i = x_{i+1}$ that each intersect the polytope in a facet. The hyperplane $x_i = 1$ intersects the polytope in a facet if one of the following three cases holds: $\mathbf{v}_{i-1} \mathbf{v}_i = \mathbf{xy}$; $i = 1$ and $\mathbf{v}_1 = \mathbf{y}$; or $i = n$ and $\mathbf{v}_n = \mathbf{x}$. A similar statement holds for the hyperplane $x_i = 0$. The lemma follows by adding these three statements. \square

For a \mathbf{xy} -word $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{n-1}$ and a subset T of $[n-1]$, define \mathbf{v}^T to be the subword $\mathbf{v}^T = \mathbf{v}_{j_1} \mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$, where $T = \{j_1 < j_2 < \cdots < j_k\}$. The following theorem provides a way to compute the f -polynomial $F_{\mathbf{v}}$.

Theorem 2.2. *Let \mathbf{v} be a \mathbf{xy} -word of length $n-1$. Then the f -polynomial of the descent polytope $\text{DP}_{\mathbf{v}}$ is given by*

$$F_{\mathbf{v}} = 1 + \sum_{T \subseteq [n-1]} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}.$$

Proof. For a face \mathcal{F} of a polytope, let \mathcal{F}^I denote the relative interior of \mathcal{F} . Then the polytope is the disjoint union of \mathcal{F}^I taken over all faces \mathcal{F} , including the polytope itself.

Recall that the descent polytope $\text{DP}_{\mathbf{v}}$ consists of all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ belonging simultaneously to the half spaces $x_i \geq 0$, $x_i \leq 1$ ($1 \leq i \leq n$), $x_i \leq x_{i+1}$ ($\mathbf{v}_i = \mathbf{x}$), and $x_i \geq x_{i+1}$ ($\mathbf{v}_i = \mathbf{y}$). A face \mathcal{F} of $\text{DP}_{\mathbf{v}}$ can be uniquely identified by specifying which of these half spaces contain \mathcal{F} on their boundary hyperplanes, as long as the intersection of the whole polytope and the specified boundary hyperplanes is non-empty. Forming the specification just for the half spaces of the form $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$ restricts the location of \mathcal{F}^I in \mathbb{R}^n to the region defined by the relations

$$x_1 = x_2 = \dots = x_{j_1} \lessgtr x_{j_1+1} = x_{j_1+2} = \dots = x_{j_2} \lessgtr \dots \lessgtr x_{j_k+1} = x_{j_k+2} = \dots = x_n \quad (2.1)$$

for some $T = \{j_1 < j_2 < \dots < j_k\} \subseteq [n-1]$, where the symbol \lessgtr denotes strict inequality: $x_{j_i} < x_{j_i+1}$ if $\mathbf{v}_{j_i} = \mathbf{x}$, or $x_{j_i} > x_{j_i+1}$ if $\mathbf{v}_{j_i} = \mathbf{y}$. Then T is the set of indexes j for which \mathcal{F} does *not* lie entirely on the boundary hyperplane $x_j = x_{j+1}$ and thus the relative interior \mathcal{F}^I is contained in the interior of the corresponding half space. Let $\mathcal{R}(T)$ denote the intersection of the region defined by (2.1) and the hypercube $[0, 1]^n$. Each point (x_1, \dots, x_n) of $\text{DP}_{\mathbf{v}}$ belongs to exactly one such region $\mathcal{R}(T)$, namely, the one for $T = \{j \mid x_j \neq x_{j+1}\}$. Thus we have the disjoint union

$$\text{DP}_{\mathbf{v}} = \bigsqcup_{T \subseteq [n-1]} \mathcal{R}(T).$$

Let us show that the term corresponding to $T \neq \emptyset$ in the expression in the statement of the theorem is the contribution to $F_{\mathbf{v}}$ of the faces \mathcal{F} of $\text{DP}_{\mathbf{v}}$ for which \mathcal{F}^I is contained in the region $\mathcal{R}(T)$. In other words, we claim that for $T \neq \emptyset$ we have

$$\sum_{\mathcal{F} : \mathcal{F}^I \subseteq \mathcal{R}(T)} t^{\dim \mathcal{F}} = \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}. \quad (2.2)$$

Fix $\emptyset \neq T \subseteq [n-1]$. To select a particular face \mathcal{F} from the set of all faces with the property $\mathcal{F}^I \subseteq \mathcal{R}(T)$, we need to complete the specification started above, that is, we must specify which of the hyperplanes $x_i = 0, 1$ contain \mathcal{F} , and we must make sure that the intersection of the set of the specified hyperplanes and $\mathcal{R}(T)$ is non-empty. In terms of defining relations (2.1), this task is equivalent to setting the common value of some of the “blocks” of coordinates (x_1, \dots, x_{j_1}) , $(x_{j_1+1}, \dots, x_{j_2})$, \dots , (x_{j_k+1}, \dots, x_n) to 0 or 1. Since the relations must remain satisfiable by at least one point in $[0, 1]^n$, only the blocks preceded in (2.1) by $>$ (or nothing) and succeeded by $<$ (or nothing) can be set to 0. Similarly, only the blocks preceded by $<$ (or nothing) and succeeded by $>$ (or nothing) can be set to 1. Thus each block can be set to at most one of 0 and 1. The letters of the \mathbf{xy} -word $\mathbf{v}^T = \mathbf{v}_{j_1} \dots \mathbf{v}_{j_k}$ encode the inequality signs in (2.1) (\mathbf{x} stands for $<$, and \mathbf{y} stands for $>$), so the number of blocks that can be set to 0 or 1 is the total number of occurrences of \mathbf{x} followed by \mathbf{y} , or \mathbf{y} followed by \mathbf{x} , in \mathbf{v}^T , plus 2, as we also need to count the first and the last blocks. In other words, the number of such blocks is $\kappa(\mathbf{v}^T)$.

Observe that the dimension of the face of $\text{DP}_{\mathbf{v}}$ obtained by this specification procedure equals the number of blocks that have not been set to 0 or 1: the common values of the coordinates in those blocks form the “degrees of freedom” that constitute the dimension. Let us call such blocks *free*. The number of faces \mathcal{F} with $\mathcal{F}^I \subseteq \mathcal{R}(T)$ for which the specification procedure results in m free blocks is

$$\binom{\kappa(\mathbf{v}^T)}{|T| + 1 - m},$$

the number of ways to choose $|T| + 1 - m$ blocks that are *not* free out of $\kappa(\mathbf{v}^T)$ possibilities. Hence we have

$$\begin{aligned} \sum_{\mathcal{F} : \mathcal{F}^T \subseteq \mathcal{R}(T)} t^{\dim \mathcal{F}} &= \sum_{m=|T|+1-\kappa(\mathbf{v}^T)}^{|T|+1} \binom{\kappa(\mathbf{v}^T)}{|T|+1-m} \cdot t^m \\ &= t^{|T|+1-\kappa(\mathbf{v}^T)} \cdot \sum_{\ell=0}^{\kappa(\mathbf{v}^T)} \binom{\kappa(\mathbf{v}^T)}{\ell} \cdot t^\ell \\ &= t^{|T|+1-\kappa(\mathbf{v}^T)} \cdot (t+1)^{\kappa(\mathbf{v}^T)}, \end{aligned}$$

proving (2.2).

Finally, for $T = \emptyset$, we have $\mathcal{R}(T) = \{0 \leq x_1 = \dots = x_n \leq 1\}$, which is just the line segment joining the two vertices $(0, \dots, 0)$ and $(1, \dots, 1)$ of $\text{DP}_{\mathbf{v}}$. Thus the contribution of $\mathcal{R}(T)$ to $F_{\mathbf{v}}$ is

$$t + 2 = 1 + \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^\emptyset)} \cdot t.$$

Adding this equation to the sum of (2.2) taken over the non-empty T proves the theorem. \square

Theorem 2.2 yields a combinatorial interpretation of the number of vertices of the polytope $\text{DP}_{\mathbf{v}}$. Call a **xy**-word $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_k$ *alternating* if $\mathbf{v}_i \neq \mathbf{v}_{i+1}$ for all $1 \leq i \leq k-1$. Then we have the following corollary.

Corollary 2.3. *For \mathbf{v} a **xy**-word of length $n-1$, the number of vertices of the descent polytope $\text{DP}_{\mathbf{v}}$ is one greater than the number of subsets $T \subseteq [n-1]$ for which the word \mathbf{v}^T is alternating.*

Proof. The number of vertices of $\text{DP}_{\mathbf{v}}$ is the constant term of $F_{\mathbf{v}}$. For the summand corresponding to a subset $T \subseteq [n-1]$ in the formula of Theorem 2.2, the constant term is either 0 or 1, the latter being the case if and only if $|T| + 1 - \kappa(\mathbf{v}^T) = 0$. This condition is equivalent to \mathbf{v}^T being alternating, proving the corollary. \square

As we mention in the introduction, the descent set statistic $\beta(S)$ is maximized when S is the alternating set. The most elegant proof of this fact uses the **cd**-index of the simplex; see [7]. For a **xy**-word \mathbf{v} , let $\bar{\mathbf{v}}$ denote the word obtained from \mathbf{v} by replacing **x**'s with **y**'s and vice versa. Then the following inequality holds:

$$\beta(\mathbf{u}\mathbf{y}\mathbf{x}\mathbf{v}) > \beta(\mathbf{u}\mathbf{y}\mathbf{y}\bar{\mathbf{v}}), \quad (2.3)$$

where we use **xy**-words to encode the sets. In each of the proofs [1, 6, 8, 11] that the alternating word maximizes the descent set, the arguments rely on proving the inequality (2.3). However, the **cd**-index proof gives a quick way to verify this inequality. We now state a similar inequality for the f -vectors of descent polytopes.

Theorem 2.4. *Let \mathbf{u} and \mathbf{v} be two **xy**-words such that $|\mathbf{u}| + |\mathbf{v}| = n-3$. Then the difference*

$$F_{\mathbf{u}\mathbf{y}\mathbf{x}\mathbf{v}}(t) - F_{\mathbf{u}\mathbf{y}\mathbf{y}\bar{\mathbf{v}}}(t) \quad (2.4)$$

has positive coefficients at $1, t, \dots, t^{n-1}$. That is, for $0 \leq i \leq n-1$ the descent polytope $\text{DP}_{\mathbf{u}\mathbf{y}\mathbf{x}\mathbf{v}}$ has more faces of dimension i than the descent polytope $\text{DP}_{\mathbf{u}\mathbf{y}\mathbf{y}\bar{\mathbf{v}}}$.

\mathbf{u}^T	\mathbf{v}^U	$\kappa(\mathbf{u}^T \bar{\mathbf{v}}^U)$	$\frac{Q_{T,U}(t)}{(t+1)^k t^{ T + U +1-k}}$	$\frac{\bar{Q}_{T,U}(t)}{(t+1)^k t^{ T + U +1-k}}$	$\frac{Q_{T,U}(t) - \bar{Q}_{T,U}(t)}{(t+1)^{k-1} t^{ T + U +1-k}}$
$\cdots \mathbf{x}$	$\mathbf{x} \cdots$	$k+1$	$1+t+(t+1)^2 t^{-1} + (t+1)^2$	$(t+1) t^{-1} \cdot (1+2t+t^2)$	$(t+1)^2$
$\cdots \mathbf{x}$	$\mathbf{y} \cdots$	$k-1$	$1+t+t + (t+1)^2$	$(t+1)^{-1} t \cdot (1+2(t+1)^2 t^{-1} + (t+1)^2)$	t^2
$\cdots \mathbf{y}$	$\mathbf{x} \cdots$	$k-1$	$1+t+t+t^2$	$(t+1)^{-1} t \cdot (1+2t+t^2)$	$(t+1)^2$
$\cdots \mathbf{y}$	$\mathbf{y} \cdots$	$k+1$	$1+(t+1)^2 t^{-1} + t + (t+1)^2$	$(t+1) t^{-1} \cdot (1+2t+t^2)$	$(t+1)^2$
1	$\mathbf{x} \cdots$	k	$1+t+(t+1) + (t+1) t$	$1+2t+t^2$	$(t+1)^2$
1	$\mathbf{y} \cdots$	k	$1+(t+1) + t + (t+1)^2$	$1+2(t+1) + (t+1) t$	$t^2 + t$
$\cdots \mathbf{x}$	1	k	$1+t+(t+1) + (t+1)^2$	$1+2(t+1) + (t+1) t$	$t^2 + t$
$\cdots \mathbf{y}$	1	k	$1+(t+1) + t + (t+1) t$	$1+2t+t^2$	$(t+1)^2$
1	1	$k=1$	$1+(t+1) + (t+1) + (t+1)^2$	$1+2(t+1) + (t+1) t$	$(t+1)^2$

Table 1: Calculations for the proof of Theorem 2.4.

Proof. Let $|\mathbf{u}| = m$ and $|\mathbf{v}| = n - m - 3$. For $T \subseteq [m]$ and $U \subseteq [n - m - 3]$, define

$$Q_{T,U}(t) := \sum_{E \subseteq \{1,2\}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{u}^T(\mathbf{y}\mathbf{x})^E \mathbf{v}^U)} \cdot t^{|T|+|U|+|E|+1}. \quad (2.5)$$

Thus $Q_{T,U}(t)$ is the sum of four of the terms in the summation formula for $F_{\mathbf{u}\mathbf{y}\mathbf{x}\mathbf{v}}(t)$ given by Theorem 2.2, corresponding to fixed choices of letters drawn from \mathbf{u} and from \mathbf{v} . Similarly, let us define

$$\overline{Q}_{T,U}(t) := \sum_{E \subseteq \{1,2\}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{u}^T(\mathbf{y}\mathbf{y})^E \overline{\mathbf{v}}^U)} \cdot t^{|T|+|U|+|E|+1}.$$

Note that $Q_{T,U}(t)$ depends only on $|T|$, $|U|$, $\kappa(\mathbf{u}^T \mathbf{v}^U)$, the last letter of \mathbf{u}^T , and the first letter of \mathbf{v}^U , and not on the remaining letters of \mathbf{u}^T and \mathbf{v}^U . Thus to show that the difference $Q_{T,U}(t) - \overline{Q}_{T,U}(t)$ is a polynomial with non-negative coefficients, it suffices to consider 9 cases corresponding to \mathbf{u}^T (respectively, \mathbf{v}^U) ending (respectively, beginning) with \mathbf{x} or \mathbf{y} , or being equal to the empty word 1.

We summarize our calculations in Table 1. We denote $\kappa(\mathbf{u}^T \mathbf{v}^U)$ by k , and we divide each polynomial by the common factor $(t+1)^k \cdot t^{|T|+|U|+1-k}$. In the fourth column, which corresponds to $Q_{T,U}$, the four summands represent the results of inserting 1, \mathbf{x} , \mathbf{y} , and $\mathbf{y}\mathbf{x}$ between \mathbf{u}^T and \mathbf{v}^U . For example, if \mathbf{u}^T ends with a \mathbf{x} and \mathbf{v}^U begins with a \mathbf{x} , then inserting \mathbf{y} increases the value of the statistic κ by 2, thus contributing a factor of

$$\left(\frac{t+1}{t} \right)^2 \cdot t = (t+1)^2 \cdot t^{-1}$$

to the corresponding term of (2.5). Similarly, the entries in the fifth column consist of a factor resulting from a different value of κ for the word $\mathbf{u}^T \overline{\mathbf{v}}^U$ times the contributions of inserting 1, \mathbf{y} (counted twice), and \mathbf{y}^2 between \mathbf{u}^T and $\overline{\mathbf{v}}^U$.

We conclude that in every case the quotient $\frac{Q_{T,U}(t) - \overline{Q}_{T,U}(t)}{(t+1)^{k-1} \cdot t^{|T|+|U|+1-k}}$ is a polynomial of degree 2 with non-negative coefficients. Hence the difference $Q_{T,U}(t) - \overline{Q}_{T,U}(t)$ is a polynomial with non-negative coefficients. Summing over all possible pairs (T, U) yields that the difference in equation (2.4) has non-negative coefficients. More specifically, the polynomial $Q_{T,U}(t) - \overline{Q}_{T,U}(t)$ has degree $(k-1) + (|T| + |U| + 1 - k) + 2 = |T| + |U| + 2$. This degree can attain any integer value between 2 and $n-1$. Thus the leading terms of these differences contribute positively to the coefficients of t^2, t^3, \dots, t^{n-1} in the difference (2.4). Furthermore, in the case $T = U = \emptyset$ we have $Q_{T,U}(t) - \overline{Q}_{T,U}(t) = (t+1)^2$, which yields a positive contribution to the constant and the linear terms of the overall difference. The proof is now complete. \square

Let \mathbf{z}_n be the alternating word of length n starting with the letter \mathbf{x} . Then $\overline{\mathbf{z}}_n$ is the alternating word beginning with \mathbf{y} . That is, the two alternating words are

$$\mathbf{z}_n = \underbrace{\mathbf{x}\mathbf{y}\mathbf{x} \cdots}_n \quad \text{and} \quad \overline{\mathbf{z}}_n = \underbrace{\mathbf{y}\mathbf{x}\mathbf{y} \cdots}_n.$$

We now have the maximization result for the f -vector of descent polytopes.

Corollary 2.5. *The f -vector of the two descent polytopes $\text{DP}_{\mathbf{z}_{n-1}}$ and $\text{DP}_{\overline{\mathbf{z}}_{n-1}}$ is maximal among the f -vectors of all descent polytopes of dimension n . That is, for each $0 \leq i \leq n-1$, the polytope $\text{DP}_{\mathbf{z}_{n-1}}$ has more faces of dimension i than the descent polytope $\text{DP}_{\mathbf{v}}$ of dimension n for an non-alternating word \mathbf{v} .*

3 The power series $\Phi(\mathbf{x}, \mathbf{y})$

We now derive a non-commutative generating function $\Phi(\mathbf{x}, \mathbf{y})$ for the f -polynomial $F_{\mathbf{v}}$, which belongs to the ring $\Phi(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}[t]\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle$. We define the power series $\Phi(\mathbf{x}, \mathbf{y})$ by

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}} \cdot \mathbf{v},$$

where the sum is over all \mathbf{xy} -words \mathbf{v} . Since we have the symmetry $F_{\mathbf{v}} = F_{\overline{\mathbf{v}}}$, we obtain that $\Phi(\mathbf{x}, \mathbf{y})$ is symmetric with respect to \mathbf{x} and \mathbf{y} , that is,

$$\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x}).$$

Let \mathbf{v} be a \mathbf{xy} -word $\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{n-1}$. Consider the following polynomials:

$$\begin{aligned} K_{\mathbf{v}}(t) &:= \sum_{T \subseteq [n-1] : \mathbf{v}_{j_1} = \mathbf{x}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}, \\ L_{\mathbf{v}}(t) &:= \sum_{T \subseteq [n-1] : \mathbf{v}_{j_1} = \mathbf{y}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^T)} \cdot t^{|T|+1}, \end{aligned}$$

where \mathbf{v}_{j_1} denotes the first letter of the word $\mathbf{v}^T = \mathbf{v}_{j_1} \mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$, as in the notation of Theorem 2.2. Since \mathbf{v}^T begins with either \mathbf{x} or \mathbf{y} unless $T = \emptyset$, we have

$$F_{\mathbf{v}} = K_{\mathbf{v}} + L_{\mathbf{v}} + t + 2. \quad (3.1)$$

We continue with a lemma that relates the two polynomials $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$.

Lemma 3.1. *For a \mathbf{xy} -word \mathbf{v} the following four equalities hold:*

$$\begin{aligned} K_{\mathbf{y}\mathbf{v}} &= K_{\mathbf{v}}, \\ L_{\mathbf{x}\mathbf{v}} &= L_{\mathbf{v}}, \\ K_{\mathbf{x}\mathbf{v}} &= L_{\mathbf{y}\mathbf{v}} = (t+1) \cdot (K_{\mathbf{v}} + L_{\mathbf{v}} + t + 1). \end{aligned}$$

Proof. For an integer i and a set $U \subseteq \mathbb{Z}$, let $U + i$ denote the set obtained by adding i to each element of U . Also let $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{n-1}$, where each \mathbf{v}_i is either \mathbf{x} or \mathbf{y} .

Clearly, $(\mathbf{y}\mathbf{v})^T$ begins with \mathbf{x} if and only if $1 \notin T$ and \mathbf{v}^{T-1} begins with \mathbf{x} , in which case $(\mathbf{y}\mathbf{v})^T = \mathbf{v}^{T-1}$. Hence $K_{\mathbf{y}\mathbf{v}} = K_{\mathbf{v}}$.

Now, $(\mathbf{x}\mathbf{v})^T$ begins with \mathbf{x} if and only if either $1 \in T$, or else $1 \notin T$ and \mathbf{v}^{T-1} begins with \mathbf{x} . In the former case, we have $T = \{1 < j_1 + 1 < j_2 + 1 < \cdots < j_k + 1\}$, and $(\mathbf{x}\mathbf{v})^T = \mathbf{x}\mathbf{v}_{j_1} \mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k}$. Set $U = (T - \{1\}) - 1 = \{j_1 < \cdots < j_k\}$. Then $\kappa((\mathbf{x}\mathbf{v})^T) = \kappa(\mathbf{v}^U)$ if $\mathbf{v}_{j_1} = \mathbf{x}$, and $\kappa((\mathbf{x}\mathbf{v})^T) = \kappa(\mathbf{v}^U) + 1$ if $\mathbf{v}_{j_1} = \mathbf{y}$. Hence

$$\begin{aligned} \sum_{1 \in T \subseteq [n]} \left(\frac{t+1}{t} \right)^{\kappa((\mathbf{x}\mathbf{v})^T)} \cdot t^{|T|+1} &= (t+1)^2 + t \cdot \sum_{U : \mathbf{v}_{j_1} = \mathbf{x}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^U)} \cdot t^{|U|+1} \\ &\quad + (t+1) \cdot \sum_{U : \mathbf{v}_{j_1} = \mathbf{y}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^U)} \cdot t^{|U|+1} \\ &= (t+1)^2 + t \cdot K_{\mathbf{v}} + (t+1) \cdot L_{\mathbf{v}}, \end{aligned} \quad (3.2)$$

where the leading term $(t+1)^2$ corresponds to $T = \{1\}$ and $U = \emptyset$. In the case where $1 \notin T$ and \mathbf{v}^{T-1} begins with \mathbf{x} we have, as before, $(\mathbf{xv})^T = \mathbf{v}_{j_1}\mathbf{v}_{j_2} \cdots \mathbf{v}_{j_k} = \mathbf{v}^{T-1}$, and hence

$$\sum_{T : \mathbf{v}_{j_1} = \mathbf{x}} \left(\frac{t+1}{t} \right)^{\kappa((\mathbf{xv})^T)} \cdot t^{|T|+1} = \sum_{\mathbf{v}_{j_1} = \mathbf{x}} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}^{T-1})} \cdot t^{|T-1|+1} = K_{\mathbf{v}}. \quad (3.3)$$

Adding (3.2) and (3.3) yields

$$K_{\mathbf{xv}} = (t+1) \cdot (K_{\mathbf{v}} + L_{\mathbf{v}} + t+1).$$

The relations for $L_{\mathbf{xv}}$ and $L_{\mathbf{yv}}$ follow from symmetry that arises from exchanging the variables \mathbf{x} and \mathbf{y} . \square

Starting with $K_1 = L_1 = 0$, one can use Lemma 3.1 to recursively compute $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$, and hence $F_{\mathbf{v}}$, from (3.1). Recall the generating power series

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{v}} F_{\mathbf{v}} \cdot \mathbf{v},$$

where the sum is over all \mathbf{xy} -words, including the empty word $\mathbf{v} = \mathbf{v}_{\emptyset} = 1$. Define the two generating power series

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}) &:= \sum_{\mathbf{v}} K_{\mathbf{v}} \cdot \mathbf{v}, \\ \Lambda(\mathbf{x}, \mathbf{y}) &:= \sum_{\mathbf{v}} L_{\mathbf{v}} \cdot \mathbf{v}. \end{aligned}$$

From the definitions of $K_{\mathbf{v}}$ and $L_{\mathbf{v}}$ it follows that $K_{\mathbf{v}} = L_{\overline{\mathbf{v}}}$. Hence we have that

$$K(\mathbf{x}, \mathbf{y}) = \overline{\Lambda(\mathbf{x}, \mathbf{y})}.$$

Then, by (3.1), we have

$$\begin{aligned} \Phi(\mathbf{x}, \mathbf{y}) &= K(\mathbf{x}, \mathbf{y}) + \Lambda(\mathbf{x}, \mathbf{y}) + (t+2) \cdot \sum_{\mathbf{v}} \mathbf{v} \\ &= K(\mathbf{x}, \mathbf{y}) + K(\mathbf{y}, \mathbf{x}) + (t+2) \cdot \sum_{r \geq 0} (\mathbf{x} + \mathbf{y})^r \\ &= K(\mathbf{x}, \mathbf{y}) + K(\mathbf{y}, \mathbf{x}) + (t+2) \cdot \frac{1}{1 - \mathbf{x} - \mathbf{y}}. \end{aligned} \quad (3.4)$$

Using the equations in Lemma 3.1 and recalling that $K_1 = 0$ we obtain

$$\begin{aligned} K(\mathbf{x}, \mathbf{y}) &= \sum_{\mathbf{v}} K_{\mathbf{xv}} \cdot \mathbf{xv} + \sum_{\mathbf{v}} K_{\mathbf{yv}} \cdot \mathbf{yv} \\ &= (t+1) \cdot \mathbf{x} \cdot \sum_{\mathbf{v}} (K_{\mathbf{v}} + L_{\mathbf{v}} + t+1) \cdot \mathbf{v} + \mathbf{y} \cdot \sum_{\mathbf{v}} K_{\mathbf{v}} \cdot \mathbf{v} \\ &= (t+1) \cdot \mathbf{x} \cdot \left(K(\mathbf{x}, \mathbf{y}) + \Lambda(\mathbf{x}, \mathbf{y}) + (t+1) \cdot \frac{1}{1 - \mathbf{x} - \mathbf{y}} \right) + \mathbf{y} \cdot K(\mathbf{x}, \mathbf{y}) \\ &= (t+1) \cdot \mathbf{x} \cdot \left(\Phi(\mathbf{x}, \mathbf{y}) - \frac{1}{1 - \mathbf{x} - \mathbf{y}} \right) + \mathbf{y} \cdot K(\mathbf{x}, \mathbf{y}), \end{aligned}$$

where the last step is by equation (3.4). Rearranging terms we have

$$K(\mathbf{x}, \mathbf{y}) = (t+1) \cdot (1-\mathbf{y})^{-1} \cdot \mathbf{x} \cdot \left(\Phi(\mathbf{x}, \mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}} \right).$$

Adding this equation and its symmetric version obtained by exchanging \mathbf{x} and \mathbf{y} one has

$$K(\mathbf{x}, \mathbf{y}) + K(\mathbf{y}, \mathbf{x}) = (t+1) \cdot ((1-\mathbf{y})^{-1} \cdot \mathbf{x} + (1-\mathbf{x})^{-1} \cdot \mathbf{y}) \cdot \left(\Phi(\mathbf{x}, \mathbf{y}) - \frac{1}{1-\mathbf{x}-\mathbf{y}} \right),$$

using the symmetry $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x})$. Now using equation (3.4) we can solve for $\Phi(\mathbf{x}, \mathbf{y})$ and arrive at the following theorem.

Theorem 3.2. *The generating power series $\Phi(\mathbf{x}, \mathbf{y})$ is given by*

$$\Phi(\mathbf{x}, \mathbf{y}) = \left(1 + \frac{t+1}{1 - (t+1) \cdot ((1-\mathbf{y})^{-1} \cdot \mathbf{x} + (1-\mathbf{x})^{-1} \cdot \mathbf{y})} \right) \cdot \frac{1}{1-\mathbf{x}-\mathbf{y}}.$$

Corollary 3.3. *For a \mathbf{xy} -word \mathbf{v} the f -vector of the descent polytope $\text{DP}_{\mathbf{v}}$ is given by the sum*

$$F_{\mathbf{v}}(t) = 1 + \sum_{(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{u}_k)} (t+1)^k,$$

where the sum ranges over all factorizations of the word $\mathbf{v} = \mathbf{u}_1 \cdots \mathbf{u}_{k-1} \cdot \mathbf{u}_k$ such that each of factors $\mathbf{u}_1, \dots, \mathbf{u}_{k-1}$ are of the form $\mathbf{x}^i \mathbf{y}$ or $\mathbf{y}^i \mathbf{x}$, where $i \geq 0$, and there is no condition on the last factor \mathbf{u}_k .

Proof. Rewrite Theorem 3.2 as

$$\begin{aligned} \Phi(\mathbf{x}, \mathbf{y}) &= \frac{1}{1-\mathbf{x}-\mathbf{y}} + \frac{1}{1 - (t+1) \cdot ((1-\mathbf{y})^{-1} \cdot \mathbf{x} + (1-\mathbf{x})^{-1} \cdot \mathbf{y})} \cdot \frac{t+1}{1-\mathbf{x}-\mathbf{y}} \\ &= \sum_{\mathbf{v}} \mathbf{v} + \sum_{j \geq 0} \left((t+1) \cdot \sum_{i \geq 0} (\mathbf{y}^i \mathbf{x} + \mathbf{x}^i \mathbf{y}) \right)^j \cdot (t+1) \cdot \sum_{\mathbf{v}} \mathbf{v}, \end{aligned}$$

where in both sums \mathbf{v} ranges over all \mathbf{xy} -words. The corollary follows by reading the generating function. \square

Example 3.4. Consider the 5-dimensional descent polytope $\text{DP}_{\mathbf{v}}$ where $\mathbf{v} = \mathbf{xyyx}$. We have the following list of 11 factorizations:

$$\begin{aligned} \mathbf{v} &= \mathbf{xyyx} &= \mathbf{x} \cdot \mathbf{yyx} &= \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{yx} &= \mathbf{xy} \cdot \mathbf{yx} \\ &= \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{y} \cdot \mathbf{x} &= \mathbf{xy} \cdot \mathbf{y} \cdot \mathbf{x} &= \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{y} \cdot \mathbf{x} \cdot 1 &= \mathbf{xy} \cdot \mathbf{y} \cdot \mathbf{x} \cdot 1 \\ &= \mathbf{x} \cdot \mathbf{y} \cdot \mathbf{yx} \cdot 1 &= \mathbf{xy} \cdot \mathbf{yx} \cdot 1 &= \mathbf{x} \cdot \mathbf{yyx} \cdot 1 \end{aligned}$$

Hence the f -polynomial of the polytope $\text{DP}_{\mathbf{xyyx}}$ is given by

$$\begin{aligned} F_{\mathbf{xyyx}} &= 1 + (t+1) + 2 \cdot (t+1)^2 + 4 \cdot (t+1)^3 + 3 \cdot (t+1)^4 + (t+1)^5 \\ &= 12 + 34 \cdot t + 42 \cdot t^2 + 26 \cdot t^3 + 8 \cdot t^4 + t^5. \end{aligned}$$

For the alternating word \mathbf{z}_{n-1} we can say more about the associated descent polytope. The number of vertexes of $\text{DP}_{\mathbf{z}_{n-1}}$ has Fibonacci number of vertexes; see for instance [10, Chapter 1, Exercise 14e]. More generally, the f -vector of $\text{DP}_{\mathbf{z}_{n-1}}$ is given by the next result.

Corollary 3.5. *The f -polynomial of the n -dimensional descent polytope $\text{DP}_{\mathbf{z}_{n-1}}$ is described by*

$$F_{\mathbf{z}_{n-1}} = 1 + \sum_{(c_1, c_2, \dots, c_k)} (t+1)^k,$$

where the sum is over all compositions of n such that all but the last part is less than or equal to 2, that is, $c_1, \dots, c_{k-1} \in \{1, 2\}$.

Proof. The only factors of the alternating word \mathbf{z}_{n-1} of the form $\mathbf{x}^i \mathbf{y}$ or $\mathbf{y}^i \mathbf{x}$ has $i = 0, 1$. Hence it is enough to record the length of each factor \mathbf{u}_i , that is, $d_i = |\mathbf{u}_i|$. Thus we are summing over vectors of non-negative integers (d_1, \dots, d_k) such that the sum of the entries is $n-1$ and $d_1, \dots, d_{k-1} \in \{1, 2\}$ and $d_k \geq 0$. By adding one to the last entry d_k we have a composition of n . \square

This corollary yields the generating function

$$\sum_{n \geq 1} F_{\mathbf{z}_{n-1}} \cdot x^n = \frac{x}{1-x} + \frac{1}{1-(t+1) \cdot (x+x^2)} \cdot (t+1) \cdot \frac{x}{1-x}. \quad (3.5)$$

Setting $t = 0$ in this generating function and adding constant 1 yields $(1+x)/(1-x-x^2)$, the generating function for the Fibonacci numbers as expected.

4 More recurrence relations

In this section we derive a different set of recurrences determining $F_S(t)$ than the ones we used to obtain Theorem 3.2. Here it will be more convenient to associate integer sets with compositions. Let $\text{Comp}'(m)$ denote the set of integer compositions $(\gamma_1, \gamma_2, \dots)$ of m with $\gamma_1 \geq 0$ and $\gamma_2, \gamma_3, \dots > 0$. For $\gamma = (\gamma_1, \gamma_2, \dots) \in \text{Comp}'(m)$, define $\mathbf{v}_\gamma = \mathbf{x}^{\gamma_1} \mathbf{y}^{\gamma_2} \mathbf{x}^{\gamma_3} \mathbf{y}^{\gamma_4} \dots$. This is a bijection between compositions in $\text{Comp}'(m)$ and \mathbf{xy} -words of length m . By composing this bijection with the correspondence between words and subsets we have a bijection between subsets of $[m]$ and $\text{Comp}'(m)$. For example, if we have $S = \{1, 3, 4\} \subseteq [6]$, then $\mathbf{v}_S = \mathbf{yxyyxx} = \mathbf{x}^0 \mathbf{y}^1 \mathbf{x}^1 \mathbf{y}^2 \mathbf{x}^2$ and thus $c(S) = (0, 1, 1, 2, 2)$.

Write $F_S(t) = F_{c(S)}(t)$. In the notation of Theorem 2.2, define

$$\begin{aligned} G_\gamma(t) &:= t+1 + \sum_{T: i_1 > \gamma_1} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}_\gamma^T)} \cdot t^{|T|+1}, \\ H_\gamma(t) &:= \sum_{T: i_1 \leq \gamma_1} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}_\gamma^T)} \cdot t^{|T|+1}, \end{aligned}$$

where $T = \{i_1 < i_2 < \dots\} \subseteq [n-1]$. Thus $F_\gamma = 1 + G_\gamma + H_\gamma$. The extra $t+1$ in G_γ corresponds to the term for $T = \emptyset$. Note that G_γ does not depend on the value of γ_1 , and that $\gamma_1 = 0$ implies $H_\gamma = 0$. For $\gamma = (\gamma_1, \gamma_2, \dots)$, write $\gamma^{(\ell)} = (\gamma_{\ell+1}, \gamma_{\ell+2}, \dots)$. Also, for a nonnegative integer r , define

$$p_r = p_r(t) := \frac{(t+1)^r - 1}{t} = [r]_{q=t+1},$$

where $[r]_q = \frac{q^r - 1}{q - 1}$ is the classical q -analog of r . Breaking up the summation formula for F_γ allows to obtain the following recurrence relations.

Lemma 4.1. *For $r \geq 0$, we have $G_{(r)} = t + 1$ and $H_{(r)} = p_r \cdot (t + 1)^2$. If $\gamma = (\gamma_1, \gamma_2, \dots)$ has at least two parts, then*

$$\begin{aligned} G_\gamma &= G_{\gamma(1)} + H_{\gamma(1)}, \\ H_\gamma &= p_{\gamma_1} \cdot \left((t + 1)^2 + t \cdot (H_{\gamma(1)} + H_{\gamma(2)} + \dots) + H_{\gamma(1)} + H_{\gamma(3)} + H_{\gamma(5)} + \dots \right). \end{aligned}$$

Proof. Comparing with Theorem 2.2, observe that $G_\gamma = F_{\gamma(1)} - 1 = G_{\gamma(1)} + H_{\gamma(1)}$.

Now consider the terms in the definition of H_γ . These terms correspond to $T \subseteq [n - 1]$ such that $T \cap [\gamma_1] \neq \emptyset$. Write $T = \{i_1 < \dots < i_\ell < j_1 < \dots < j_{k-\ell}\}$, where $i_\ell \leq \gamma_1$ and $j_1 > \gamma_1$. Let $I = \{i_1 < \dots < i_\ell\}$ and $J = \{j_1 < \dots < j_{k-\ell}\}$, so that $\mathbf{v}_\gamma^T = \mathbf{v}_\gamma^I \mathbf{v}_\gamma^J$. Since \mathbf{v}_γ^I is a positive power of \mathbf{x} , we have $\kappa(\mathbf{v}_\gamma^T) = \kappa(\mathbf{v}_\gamma^J)$ if \mathbf{v}_γ^J begins with a \mathbf{x} , and $\kappa(\mathbf{v}_\gamma^T) = \kappa(\mathbf{v}_\gamma^J) + 1$ if \mathbf{v}_γ^J begins with a \mathbf{y} or if $J = \emptyset$. Hence we have

$$H_\gamma = \left(\sum_{\emptyset \neq I \subseteq [\gamma_1]} t^{|I|} \right) \cdot \left(\sum_{J \subseteq [n-1] \setminus [\gamma_1]} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}_\gamma^J) + \epsilon_\gamma^J} \cdot t^{|J|+1} \right),$$

where ϵ_γ^J is 0 if \mathbf{v}_γ^J begins with a \mathbf{x} , and 1 otherwise. The sum on the left is $(t + 1)^{\gamma_1} - 1 = t \cdot p_{\gamma_1}$. Therefore

$$\begin{aligned} H_\gamma &= t \cdot p_{\gamma_1} \cdot \left(\sum_J \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}_\gamma^J)} \cdot t^{|J|+1} \right. \\ &\quad \left. + \frac{1}{t} \cdot \sum_{J : \epsilon_\gamma^J = 1} \left(\frac{t+1}{t} \right)^{\kappa(\mathbf{v}_\gamma^J)} \cdot t^{|J|+1} \right) \\ &= t \cdot p_{\gamma_1} \cdot \left(H_{\gamma(1)} + H_{\gamma(2)} + \dots + t + 1 \right. \\ &\quad \left. + \frac{1}{t} \cdot \left(H_{\gamma(1)} + H_{\gamma(3)} + H_{\gamma(5)} + \dots + t + 1 \right) \right), \end{aligned} \tag{4.1}$$

because $H_{\gamma(i)}$ is the sum of $((t + 1)/t)^{\kappa(\mathbf{v}_\gamma^J)} \cdot t^{|J|+1}$ taken over J with $\gamma_1 + \dots + \gamma_i < j_1 \leq \gamma_1 + \dots + \gamma_{i+1}$, and $\epsilon_\gamma^J = 1$ if and only if this condition on y_1 holds for odd i (or if $J = \emptyset$, which is accounted for by the two $t + 1$ terms). \square

The next lemma provides a more concise recurrence relation for H_γ .

Lemma 4.2. *For a composition $\gamma = (\gamma_1, \gamma_2, \dots)$ with at least two parts, the following equality holds:*

$$H_\gamma + H_{(\gamma_1, \gamma_3, \gamma_4, \dots)} = p_{\gamma_1} \cdot \left(t \cdot G_{\gamma(1)} + (t + 1) \cdot (1 + G_\gamma) \right).$$

Proof. Observe that, by Lemma 4.1,

$$G_\gamma = G_{\gamma(1)} + H_{\gamma(1)} = G_{\gamma(2)} + H_{\gamma(1)} + H_{\gamma(2)} = \cdots = t + 1 + \left(H_{\gamma(1)} + H_{\gamma(2)} + \cdots \right)$$

since $G_{(0)} = t + 1$. Apply the relation (4.1) to H_γ and $H_{(\gamma_1, \gamma_3, \gamma_4, \dots)}$, and then add the two resulting equations:

$$\begin{aligned} H_\gamma + H_{(\gamma_1, \gamma_3, \gamma_4, \dots)} &= t \cdot p_{\gamma_1} \cdot \left(G_\gamma + \frac{1}{t} \cdot \left(H_{\gamma(1)} + H_{\gamma(3)} + H_{\gamma(5)} + \cdots + t + 1 \right) \right) \\ &= + t \cdot p_{\gamma_1} \cdot \left(G_{\gamma(1)} + \frac{1}{t} \cdot \left(H_{\gamma(2)} + H_{\gamma(4)} + H_{\gamma(6)} + \cdots + t + 1 \right) \right) \\ &= t \cdot p_{\gamma_1} \cdot \left(G_\gamma + G_{\gamma(1)} + \frac{1}{t} \cdot (G_\gamma + t + 1) \right). \end{aligned}$$

The lemma follows. \square

A useful consequence of Lemma 4.1 is that in working with G_γ and H_γ we can concentrate on just the compositions with the first part equal to 1. Specifically, we have the following corollary.

Corollary 4.3. *The polynomials G and H satisfy*

$$\begin{aligned} G_{(\gamma_1, \gamma_2, \dots)} &= G_{(1, \gamma_2, \gamma_3, \dots)}, \\ H_{(\gamma_1, \gamma_2, \dots)} &= p_{\gamma_1} \cdot H_{(1, \gamma_2, \gamma_3, \dots)}. \end{aligned}$$

Proof. The first identity follows from an earlier observation that G_γ is independent of the first part of γ , and the second one follows from Lemma 4.1 since $p_1 = 1$. \square

Thus for a composition γ with k parts, we can compute F_γ by applying the recurrence relations of Lemmas 4.1 and 4.2 k times. For instance, to compute $F_{(2,4,3)}$ we proceed as follows:

$$\begin{aligned} G_{(1)} = G_{(3)} &= t + 1, \\ H_{(1)} &= (t + 1)^2, \\ G_{(1,3)} = G_{(4,3)} &= G_{(3)} + H_{(3)} = G_{(1)} + p_3 \cdot H_{(1)}, \\ H_{(1,3)} &= -H_{(1)} + (t \cdot G_{(3)} + (t + 1) \cdot (1 + G_{(1,3)})), \\ G_{(1,4,3)} = G_{(2,4,3)} &= G_{(4,3)} + H_{(4,3)} = G_{(1,3)} + p_4 \cdot H_{(1,3)}, \\ H_{(1,4,3)} &= -H_{(1,3)} + (t \cdot G_{(4,3)} + (t + 1) \cdot (1 + G_{(1,4,3)})), \end{aligned}$$

and finally

$$F_{(2,4,3)} = 1 + G_{(2,4,3)} + H_{(2,4,3)} = 1 + G_{(1,4,3)} + p_2 \cdot H_{(1,4,3)}.$$

The special case of the alternating word \mathbf{z}_{n-1} corresponds to the composition $\gamma = 1^{n-1} = (1, 1, \dots, 1) \models n - 1$. The generating function for $F_{1^{n-1}}(t) = F_{\mathbf{z}_{n-1}}(t)$ in equation (3.5) can be obtained from the recurrences of this section. From Lemmas 4.1 and 4.2 we get the relations

$$\begin{aligned} G_{1^{n-1}} &= G_{1^{n-2}} + H_{1^{n-2}}, \\ H_{1^{n-1}} + H_{1^{n-2}} &= t \cdot G_{1^{n-2}} + (t + 1) \cdot (1 + G_{1^{n-1}}) \end{aligned}$$

for $n \geq 2$, where we set $G_{1^0} = 0$ and $H_{1^0} = t + 1$ for convenience (it can be easily seen that the relations are valid for $n = 2$). Then we multiply the above equations by x^n and sum over all $n \geq 2$ to obtain the system of equations

$$\begin{cases} G &= x \cdot (G + H), \\ H + x \cdot H &= t \cdot x \cdot G + (t + 1) \cdot (x^2 \cdot (1 - x)^{-1} + G) + (t + 1) \cdot x, \end{cases}$$

where $G = G(t, x) := \sum_{n \geq 1} G_{1^{n-1}}(t) x^n$ and $H = H(t, x) := \sum_{n \geq 1} H_{1^{n-1}}(t) x^n$. Solving this system for G and H , we get the generating function in equation (3.5).

5 Concluding remarks

A more general invariant to study of the descent polytopes is the flag f -vector. The flag f -vector is efficiently encoded by the **cd**-index. Is there a way to describe the **cd**-index of the descent polytope in terms of the **xy**-word **u**?

Setting $t = 1$ in the polynomial $F_{\mathbf{v}}(t)$ we obtain the number of faces of the descent polytope $\text{DP}_{\mathbf{v}}$. Especially, for the alternating word \mathbf{z}_n we obtain the sequence $\{F_{\mathbf{z}_{n-1}}(1)\}_{n \geq 1} = 3, 7, 19, 51, \dots$. This sequence has a different combinatorial interpretation, as it matches the sequence A052948 in the Online Encyclopedia of Integer Sequences [5] defined as the number of paths from $(0, 0)$ to $(n + 1, 0)$ with allowed steps $(1, 1)$, $(1, 0)$ and $(1, -1)$ contained within the region $-2 \leq y \leq 2$. The generating function

$$\frac{1 - 2x^2}{1 - 3x + 2x^3}$$

given in [5] indeed results if $t = 1$ is substituted into (3.5) and the constant 1 is added. Is there a bijective proof? A first step to find such a bijective proof would be to find a statistic on these lattice paths with the same distribution as the dimensions of the faces of the descent polytope $\text{DP}_{\mathbf{z}_{n-1}}$.

For a **xy**-word $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_n$ let \mathbf{v}^* denote the reverse of the word, that is, $\mathbf{v}^* = \mathbf{v}_n \dots \mathbf{v}_2 \mathbf{v}_1$. Note that the two descent polytopes $\text{DP}_{\mathbf{v}}$ and $\text{DP}_{\mathbf{v}^*}$ only differ by a linear transformation and hence their f -polynomials agree, that is, $F_{\mathbf{v}} = F_{\mathbf{v}^*}$. However the expressions for the f -polynomials for $F_{\mathbf{v}}$ and $F_{\mathbf{v}^*}$ in Corollary 3.3 differ. Is there a bijection between the factorizations of \mathbf{v} and \mathbf{v}^* ? The number of factorizations of \mathbf{v} is also equal the number of alternating subwords of \mathbf{v} ; see Corollary 2.3. This fact also asks for a bijective proof.

More inequalities for the descent statistic has been proved in [3, 4]. Can these inequalities be extended to the f -polynomial $F_{\mathbf{v}}$? For instance, Ira Gessel asked the following question: where does the maximum of the descent set statistic occur when restricting to words \mathbf{v} of length $n - 1$ having exactly k runs of **x**'s and **y**'s. He conjectured and it was proved in [4] that the maximum occurs at the composition $(r, \underbrace{r + 1, \dots, r + 1}_a, r, \dots, r)$ where $r = \lfloor (n - 1)/k \rfloor$ and $a = (n - 1) - r \cdot k$. Would the f -polynomial be maximized at the same composition?

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